Anisotropic Homogeneous Turbulence: hierarchy and intermittency of scaling exponents in the anisotropic sectors.

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We present the first measurements of anisotropic statistical fluctuations in *perfectly* homogeneous turbulent flows. We address both problems of intermittency in anisotropic sectors and hierarchical ordering of anisotropies on a direct numerical simulation of a three dimensional *random* Kolmogorov flow. We achieved an homogeneous and anisotropic statistical ensemble by randomly shifting the forcing phases. We observe high intermittency as a function of the order of the velocity correlation within each fixed anisotropic sector and a hierarchical organization of scaling exponents at fixed order of the velocity correlation at changing the anisotropic sector.

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At the basis of Kolmogorov 1941 theory there is the idea of restoring of universality and isotropy at small scales in turbulent flows. Memory of large scale anisotropic forcing and/or boundary conditions should be quickly lost during the process of energy transfer toward small scales. The overall result being a local recovering of isotropy and universality for turbulent fluctuations at small enough scales and large enough Reynolds numbers.

In recent years, a quantitative investigation of restoring of isotropy in experimental anisotropic turbulence [1,2], numerical homogeneous shear flows [3,4] and numerical channel flows [5] questioned the main Kolmogorov paradigm, speaking explicitly of *persistence of anisotropies*. Some theoretical work has also been done [6] in order to understand how to properly link the invariance under rotation (SO(3) symmetry group) of the Navier-Stokes equations and the analysis of anisotropic fluctuations of velocity turbulence correlations.

The observed anisotropic effects in small scales turbulence is both a *theoretical* challenge and a very actual *practical* problem, opening the question whether any, realistic, anisotropic turbulent flows can ever possess statistical features independent of the (anisotropic) boundary and forcing effects. This goes under the name of universality.

Neglected anisotropic effects in high Reynolds numbers flows have also been proposed to be at the origin of different statistical properties measured for transversal and longitudinal velocity fluctuations [7]. Importance of properly disentangling isotropic and anisotropic fluctuations has also been demonstrated in the analysis of intermittency in channel flow turbulence [8].

Important step forward in the analysis of anisotropic fluctuations have recently been done in Kraichnan models, i.e. passive scalars/vectors advected by isotropic, Gaussian and white-in-time velocity field with large scale anisotropic forcing [9–14].

In those models, anomalous scaling arises as the results of a non-trivial null-space structure of the advecting operator. In these cases, correlation functions in different sectors of the rotational group show different scaling properties. Scaling exponents are universal: they do not depend on the actual value of forcing and boundary conditions, and they are fully characterized by the order of the anisotropy. Non universal effects are felt only in coefficients multiplying the power laws. Coefficients are fixed, in principle, by requiring matching with non-universal boundaries conditions in the large-scale region. Similar problems, like the very existence of scaling laws in the anisotropic sectors and, if any, what are the values of the scaling exponents and what is the dependency from universal/non-universal effects are at the forefront of experimental, numerical and theoretical research in true turbulent flows. Only a few indirect experimental investigation of scaling in different sectors [15,16] and direct decomposition in Channel flow simulations [8,17,5] have, at the moment, been attempted.

The situations is still unclear, evidences of a clear improving of scaling laws by isolating the isotropic sector have been reported, supporting the idea that the undecomposed correlations are strongly affected by the superposition of isotropic and anisotropic fluctuations [8]. Preliminary evidences of the existence of a scaling law also in the sectors with total angular momentum $j = 2$ have been reported [15,16]. The value of the exponent for the second order correlation function being close to the dimensional estimates $\xi_2^{j=2} = 4/3$, [18].

All these preliminaries investigation in real turbulent flows are flawed by the contemporary presence of anisotropies and strong non-homogeneities. The very existence of scaling laws in presence of strong non homogeneous effects can be doubted. SO(3) decomposition becomes soon intractable as soon as non-homogeneous effects cannot be neglected [6]. Moreover, in many experimental situation, anisotropies are introduced by a shear forcing coupled to all turbulent scales: something which prevents the possibility to study "pure" inertial physics. To overcome this problems we performed the first numerical investigation of a turbulent flow with strong

anisotropic forcing confined to large scales and *perfectly homogeneous* on a numerical resolution 128^3 and 256^3. We studied a fully periodic Kolmogorov flow with random, delta correlated in time, forcing phases, which we decide to call a "Random-Kolmogorov Flow" (RKF).
In this letter we present direct measurement of scaling exponents in sectors up to total angular momentum $j = 6$. Our main results support the existence of a hierarchical organization of exponents, i.e. continuous increase of exponents as a function of j. We also found a much stronger intermittency in the anisotropic sectors than in the isotropic one. We conclude with a few comments and proposals for further work in the field. Let us begin to expose a few technical details on the simulations. We performed a direct numerical simulation of a fully periodic flow with anisotropic large scales forcing. In details, we have chosen a random forcing pointing only in one direction, the z axis, with spatial dependency on the $\hat{x}$ direction only on two wavenumbers $\boldsymbol{k}_1 = (1,0,0), \boldsymbol{k}_2 = (2,0,0)$. Namely: $\boldsymbol{f}_i(\boldsymbol{k}_{1,2}) = \delta_{i,3} f_{1,2} \exp(i\theta_{1,2})$ where f_1, f_2 are two constant amplitude and θ_1, θ_2 are two random phases, delta correlated in time. The random phases allows for a homogeneous statistics also in the -otherwise- non-homogeneous direction spanned by the two wavenumbers, i.e. we have instantaneously a large scale non-homogeneity in the x direction which is averaged out by the time evolution thanks to the random re-shuffling of the forcing phases. We studied the RKF at resolution 128^3 and 256^3, we collected up to 200 eddy turn over times for the smallest resolution and up to 50 eddy turn over times for the largest resolution. Such a long averaging is necessary because as in any strongly anisotropic flow we observe the formation of persistent large scale structures inducing strong oscillation of the mean energy in time.

In Fig. 1 we show, for example, a typical time evolution for the total energy and total energy dissipation in our runs, it is interesting to notice how the high frequencies oscillation at large scales (total energy) induces by the random forcing are completely absent at small scales (energy dissipation).
In order to increase the scaling range extension we have used an hyperviscosity with a squared laplacian. Inset of Fig. 1 quantifies our degree of homogeneity. We have a high degree of homogeneity (more than 95%) in the two transverse directions, $\hat{y}, \hat{z}$, while we still observe small oscillations in the $\hat{x}$ directions (of the order of 10%); these oscillations are due to statistical fluctuations, they must averaged out in the limit of infinite statistics.
Let us now discuss the SO(3) decomposition of longitudinal structure functions:

$$S_p(\boldsymbol{r}) = \langle\, [(\boldsymbol{v}(\boldsymbol{x}) - \boldsymbol{v}(\boldsymbol{x}+\boldsymbol{r})) \cdot \hat{\boldsymbol{r}}]^p\, \rangle, \qquad (1)$$

where we have kept only the dependency on $\boldsymbol{r}$ neglecting the small non homogeneous fluctuations. We expect that the undecomposed structure functions are not the "scaling" bricks in the theory. Theoretical and numerical analysis showed that one must first decompose the structure functions on the irreducible representations of the rotational group and than asking about the scaling behavior of the projection. In practice, being the longitudinal structure functions scalar objects, their decompositions reduces to the projections on the spherical harmonics:

$$S_p(\boldsymbol{r}) = \sum_{j=0}^{\infty} \sum_{m=-j}^{j} S_p^{jm}(|r|) Y_{jm}(\hat{\boldsymbol{r}}). \qquad (2)$$

Where we have used the indeces j, m to label the total angular momentum and its projection on a reference axis, say $\hat{z}$, respectively. The whole physical information is hidden in the coefficients $S_p^{jm}(|r|)$. In particular, the main question we want to address here concerns their scaling properties: $S_p^{jm}(|r|) \sim A_{jm}|r|^{\xi^j(p)}$ and (in the case) what one can say about the values of the scaling exponents, and on their robustness against large scale physics (universality issue). Theoretical arguments suggest that if scaling exponents exist they depend only on the j eigenvalue [19]. If true turbulence follows the Kraichnan models behavior, we should expect universality of the scaling exponents (independence of large-scale boundaries), no saturation of the hierarchy ($\xi^j(p) < \xi^{j'}(p)$ if $j < j'$) and strong non universalities in the prefactors A_{jm}.
We first present in Fig. 2 results concerning the isotropic sector, $j = 0, m = 0$, comparing the undecomposed structure functions in the three direction with the projection $S_p^{00}(|r|)$ and their logarithmic local slopes (inset). Only for the projected correlation it is possible to measure (5% of accuracy) the scaling exponents by a direct log-log fit versus the scale separation, $|\boldsymbol{r}|$. The best fit gives $\xi^{j=0}(2) = 0.70 \pm 0.03$. The undecomposed structure functions are overwhelmed by anisotropic effects at all scales which spoil completely the scaling behavior.

In Fig. 3 we present an overview of all sectors j, m which have a signal-to-noise ratio high enough to ensure stable results [20]. Sectors with odd js are absent due to the parity symmetry of our observable. We measure anisotropic fluctuations up to $j = 6$. We notice from Fig. 3 a clear foliation in terms of the j index: sectors with the same j but different ms behaves very similarly. In Table 1 we present a more quantitative analysis by showing the results for the best power law fit for structure functions of orders $p = 2, 4$. The first result we want to notice is the absence of any saturation for the exponents as a function of the j value. Unfortunatly the presence of an oscillation in all $j = 2$ sectors prevents us from measuring with accuracy the exponents in this sector, we therefore refrain from giving any number in this case.

Let us also notice that the values for $j = 4$ and $j = 6$ are different from what one would have expected if the

anisotropic effects would be given by simple smooth large scale fluctuations (see Table 1). This fact leads to the conclusion that anisotropies are certainly the results of non-linear interactions in our flows, whether they corresponds to "homogeneous" fluctuations like in the Kraichnan models or to some dimensional balancing between the non-linear terms and the forcing term is still an open question. The presence of a hierarchical monotonic increasing of exponents at fixing p and changing j leads to the possible breaking of the locality assumption in high enough j sectors [21]. For locality here we mean the fact that all integrals of pressure-velocity correlation functions are convergent both in the IR and in the UV limits.

Let us conclude by assessing also the important point connected to the existence of intermittency in higher j sectors. From Table 1 we see that already for the $j = 4$ sector, and even more for $j = 6$, the fourth order scaling exponents are "almost" saturated, i.e. very close to the values of the second order exponents. It is hard to say how much such a result is a quantitative sign of strong intermittency, due to the fact that we lack a clear unambiguous dimensional –non intermittent– prediction for anisotropic exponents (see below). A fast saturation of exponents within each sector as a function of the order of the moment must somehow be expected. We imagine the statistics in the anisotropic sectors being strongly dominated by "persistent" large scale structures, introducing cliffs structures (statistically speaking) characteristic also of saturation of exponents in anisotropic scalar advection [22,23]. Saturation, in the anisotropic sectors as a function of the order of the observed moment, p, may also lead to the appearance of "persistency of anisotropies" even in presence of the observed strict hierarchical ordering, ($\xi^j(p) < \xi^{j'}(p)$ if $j < j'$) as remarked in [5]. In conclusion we have presented the first numerical exploration of an anisotropic homogeneous turbulent flow. We have confirmed that by decomposing longitudinal structure functions in terms of the eigenvectors of the rotational operator we have a dramatic improvement of the scaling behavior in the isotropic sector. We have also used the SO(3) decomposition in order to asses two important questions opened in the field of anisotropic turbulence: (i) the presence of a hierarchical organization of turbulent fluctuations as a functions of the degree of anisotropy labeled by the j index (ii) the existence of intermittency (saturation as a function of the structure function order) in anisotropic sectors. The numerically measured values for the scaling exponents $\xi^j(p)$ are not consistent with a simple "smooth" hypothesis for the nature of anisotropic fluctuations. More work is needed in order to understand the universality degree of our results as a function of the anisotropic properties of the large scale forcing. More work will be also devoted to measure fully tensorial quantities like $D_{ij}(\boldsymbol{r}) = \langle (v_i(\boldsymbol{x}) - v_i(\boldsymbol{x}+\boldsymbol{r}))(v_j(\boldsymbol{x}) - v_j(\boldsymbol{x}+\boldsymbol{r}))\rangle$ in order to be able to probe also odds sectors of the SO(3) group.

We conclude by noticing that dimensional prediction for the $\xi^j(p)$ with $j > 0$ are far from being trivial. Indeed, different dimensionless quantities can be built by using some anisotropic mean observable (the mean shear for example, or the mean squared shear in our RKF) and the energy dissipation. Dimensional predictions than, would depend on the requirement that the anisotropic correction is (or is not) an analytical, smooth deviation from the isotropic sector.

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tions This is because the non-linear inertial operator gouverning the correlation function evolution depends only on the j eigenvalue (see [6] for more details). The scaling exponents may depend on m if they are fixed by a matching with the forcing non-homogeneous term.

[20] Some sectors in Fig. 3 are absent due to strong oscillation in the projected structure functions $S_n^{jm}(|r|)$. These oscillations are probably connected to strong large scale effects felt by those particular spherical harmonics. We do not expect robustness in these cases, different forcing and/or boundary conditions may remove the oscillatory behavior.

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(j,m)	(0,0)	(4,0)	(4,2)	(6,0)	(6,2)
$\xi_2\vert\xi_2^S$	$0.70 \pm 0.03 \mid 2$	$1.67 \pm 0.07 \mid 2$	$1.7 \pm 0.1 \mid 2$	$3.4 \pm 0.2 \mid 4$	$3.3 \pm 0.2 \mid 4$
$\xi_4\vert\xi_4^S$	$1.28 \pm 0.05 \mid 4$	$2.15 \pm 0.1 \mid 4$	$2.2 \pm 0.1 \mid 4$	$3.2 \pm 0.2 \mid 4$	$3.2 \pm 0.2 \mid 4$

TABLE I. Best fit of the scaling exponents in all stable sectors. For comparison we also give, ξ_p^S, the exponents for the case of a smooth (many times differentiable) anisotropic field. Some sectors are absent due either to the small signal-to-noise ratio or to the presence of sign changes in $S_p^{jm}(|\boldsymbol{r}|)$ which prevent the very definition of a slope. Errors are estimates from the fluctuation of the logarithmic local slopes at resolution 256^3.

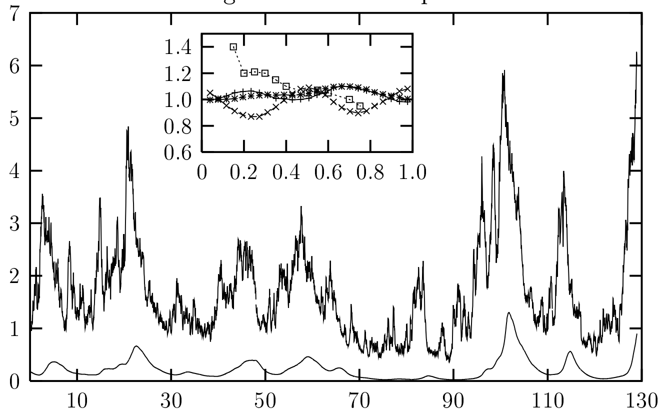

FIG. 1. Typical energy (above) and energy dissipation (below) time evolution in arbitrary units of the Random-Kolmogorov flow at resolution $L_x = L_y = L_z = 256$. Inset: root mean squared velocity $\langle v_x^2 \rangle$ as a function of the spatial location in the three directions: $\langle v_x^2(x/L_x)\rangle$ ($\times$), $\langle v_x^2(y/L_y)\rangle$ ($\star$), $\langle v_x^2(z/L_z)\rangle$ ($+$). For comparison is also shown the same quantity ($\square$) from experimental state-of-the-art anisotropic homogeneous shear flow at changing the position along the shear direction $\hat{y}$ [2]. All curves are normalized to be 1 at $x/L_x = y/L_y = z/L_z = 0.5$.

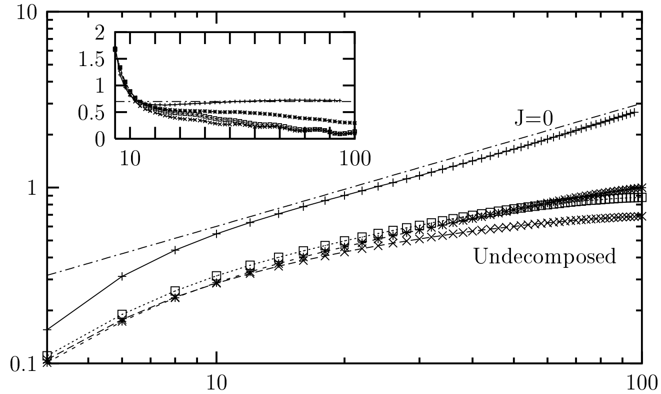

FIG. 2. Isotropic sector. Log-log plot of $S_2^{0,0}(|r|)$ versus $|r|$ ($+$), and the three undecomposed longitudinal structure functions in the three directions x, y, z ($\square, \star, \times$) respectively, at resolution 256^3. The straight line has the best fit slope $\xi^{j=0}(2) = 0.70$. Inset: logarithmic local slopes of all curves (same symbols) plus the straight line corresponding to the intermittent isotropic scaling. Notice the dramatic improvement in the scaling behavior of the projected correlation. Similar results hold for higher orders $p > 2$ (not shown).

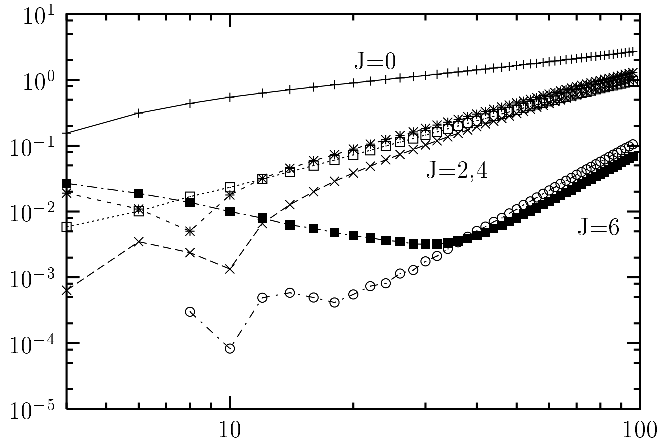

FIG. 3. Log-log plot for the absolute value of the projected second order structure functions, $|S_2^{j,m}(|r|)|$, versus the scale r, on all measurable sectors (up to $j = 6$). Sectors: $(0,0)$, $(+)$; $(2,2)$, $(\times)$; $(4,0)$, $(\square)$; $(4,2)$, $(\star)$; $(6,0)$, $(\circ)$; $(6,2)$, $(\blacksquare)$. The statistical and numerical noise induced by the SO(3) projection can be estimate as the threshold where the $j = 6$ sector starts to deviate from the monotonic decreasing behavior, i.e. $O(10^{-3})$.